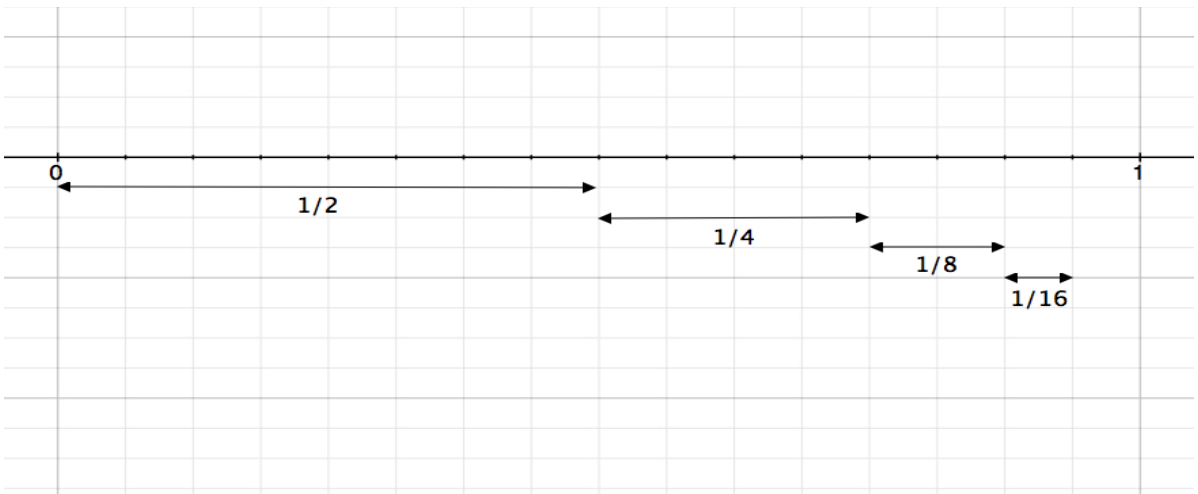
Concept of the Limit of a Function

Moving from zero to one on the real number line

Suppose I want to get from 0 to 1 on the real number line. I devise a scheme in which each time I move toward 1 from 0, I travel $\frac{1}{2}$ the distance remaining. See picture.



So far I have only gone $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}$ units from zero. But, I am not at one.

One way to look at this is to say that I can get as close to one as I like by repeating this procedure a large number of times.

You would say that I am approaching the number one. This number that we are approaching is called the **limit**.

Example involving the limit of a function.

A used car salesman earns a monthly salary of \$1200.00 and a commission of 10% on the amount of sales they generate in a given month.

We can describe the salesman's monthly earnings as a function of the amount they generate in sales using the equation,

$$y = 0.1x + 1200$$

where y is the monthly earnings and x is the amount in sales.

Let's say we want to know what value the monthly earnings will approach as the amount generated in sales approaches $75,000. We can make a table of values and investigate:

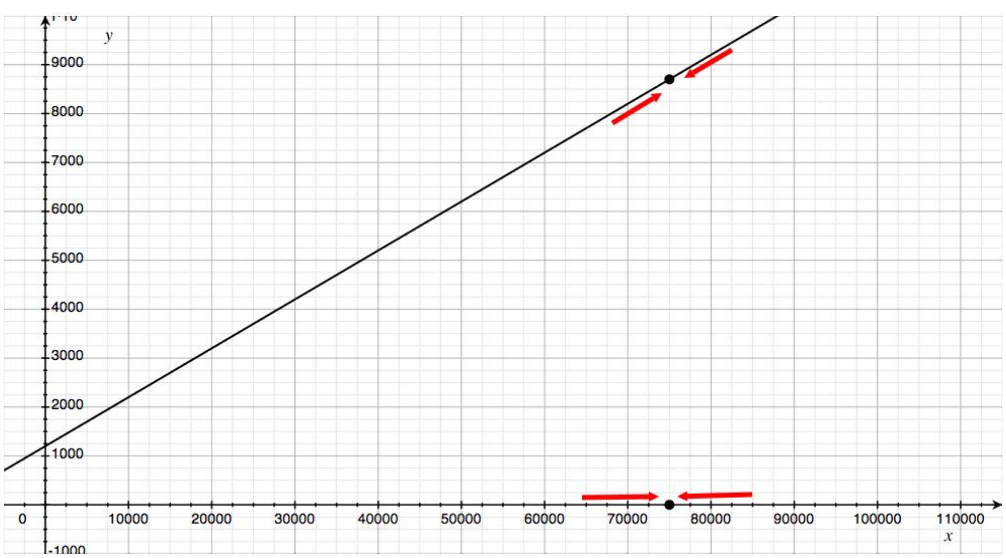
x =amount generated from sales that month	$y = 0.1x + 1200$ amount earned in that month
0	$0.1(0) + 1200 = 1200$
25,000	$0.1(25,000) + 1200 = 3700$
50,000	$0.1(50,000) + 1200 = 6200$
70,000	$0.1(70,000) + 1200 = 8200$
74,000	$0.1(74,000) + 1200 = 8600$
74,900	$0.1(74,900) + 1200 = 8690$
74,999	$0.1(74,999) + 1200 = 8699.99$
75,001	$0.1(75,001) + 1200 = 8700.10$
75,100	$0.1(75,100) + 1200 = 8710$
76,000	$0.1(76,000) + 1200 = 8800$

Can you guess what numbers should go in the blank line above?

We see that as the values of x approach 75,000, the values of y approach 8,700.

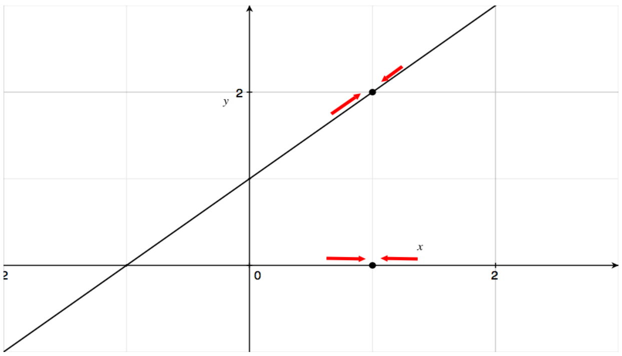
Symbolically we write,

$$\lim_{x \to 75,000} 0.1x + 1200 = 8700$$



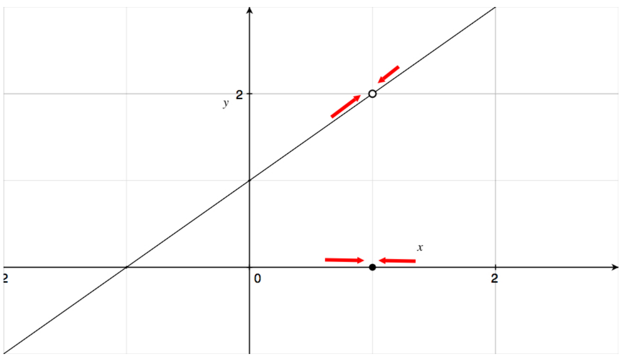
More examples involving the limit of a function.

ex.1 $y = x + 1$, The identity function shifted up one unit.



$\lim_{x \to 1} x + 1 = 2$

ex.2 $y = \dfrac{x^2 - 1}{x - 1}$, This is a rational function whose graph has a hole at $x = 1$

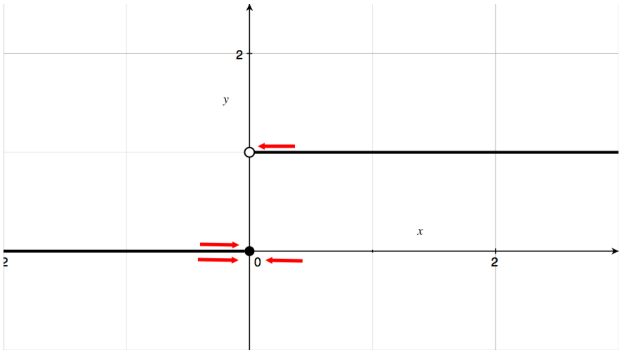


$\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$

When looking at $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$ we are asking the question, "what are the values of the function $y = \dfrac{x^2 - 1}{x - 1}$ getting close to when the values of x are getting close to 1?".

We do not necessarily care what is happening at $x = 1$, only what is happening when x is close to 1.

ex. 3 $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$, a unit step function. This function has a jump at $x = 0$.

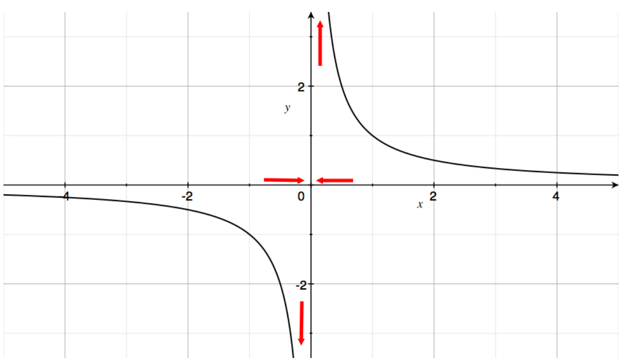


When $x \to 0$ from the right, the values of the function are approaching 1.

When $x \to 0$ from the left, the values of the function are approaching 0.

Since the values of the function are not approaching a single value as $x \to 0$ we say that **the limit does not exist**.

ex.4 $f(x) = \dfrac{1}{x}$, the reciprocal function. This function has a vertical asymptote at $x = 0$

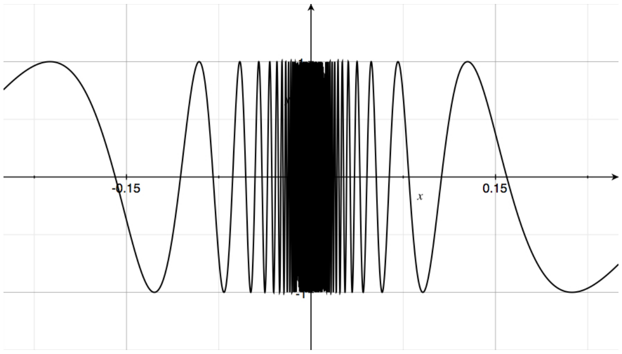


When $x \to 0$ from the right, the values of the function are growing without bound (i.e. $f(x) \to +\infty$).

When $x \to 0$ from the left, the values of the function are decreasing without bound (i.e. $f(x) \to -\infty$).

Therefore, $\lim_{x \to 0} \dfrac{1}{x}$ does not exist.

ex.5 $f(x) = \sin\left(\frac{1}{x}\right)$. This function has "too many wiggles" near $x = 0$

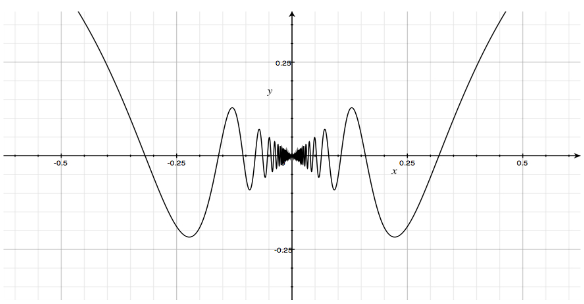


This function assumes the values of -1 and $+1$ infinitely often near $x = 0$.

Since the function values are not approaching a single number, the limit does not exist as $x \to 0$.

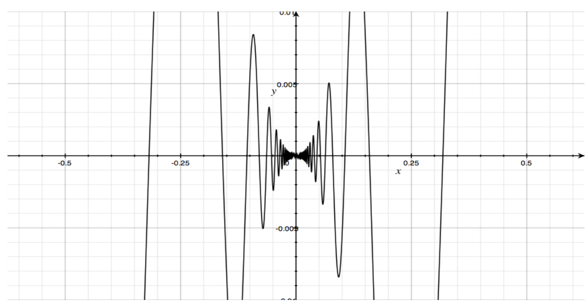
Although the function above does not approach a limiting value as $x \to 0$, the following related functions do approach a limiting value as $x \to 0$:

$f(x) = x \sin\left(\frac{1}{x}\right)$



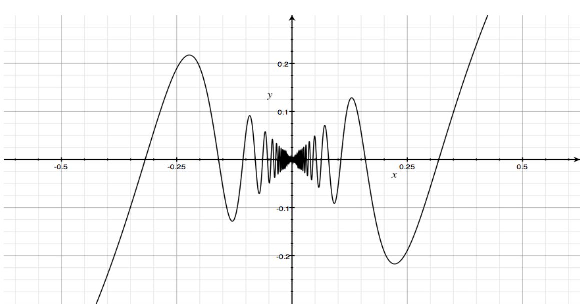
$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

$f(x) = x^2 \sin\left(\frac{1}{x}\right)$



$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

$f(x) = |x| \sin\left(\frac{1}{x}\right)$



$$\lim_{x \to 0} |x| \sin\left(\frac{1}{x}\right) = 0$$

Techniques for finding the limit of a function if it exist

1. We can show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ using the **Squeeze Theorem** (also called the Sandwich Theorem).

The idea is to get the values of the function you are working on between two other functions whose values approach a limit as $x \to 0$.

Start by noticing that,

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

We can multiply the inequality by x^2 (which is positive) and get,

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Since $-x^2 \to 0$ as $x \to 0$ and $x^2 \to 0$ as $x \to 0$, we conclude that $x^2 \sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0$

Exercises

Use the Squeeze theorem to show that:

a. $\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$

b. $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$

2. Limit Laws

a. The limit of a sum is the sum of the limits, provided the limits exist:

$$\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

b. The limit of a product is the product of the limits, provided the limits exist:

$$\lim_{x\to c}[f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$

c. The limit of a constant times a function is the constant times the limit of the function, provided the limit exists:

$$\lim_{x\to c} k \cdot f(x) = k \cdot \lim_{x\to c} f(x)$$

d. The limit of the quotient of two functions is the quotient of the limit of the functions, provided the limit exists and the limit of the denominator is not zero:

$$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

e. Raising a function to a power then taking the limit is the same as taking the limit of the function then raising it to a power:

$$\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n$$

f. This one we can not do without...

$$\lim_{x\to c} x = c$$

g. The limit of a constant is that constant

$$\lim_{x\to c} k = k$$

where k is constant.

3. Substitution. Limits of polynomials can be found by substitution and limits of rational functions can be found by substitution if the limit of the denominator is not zero.

ex.1 $\lim_{x\to 2} x^3 + 5x^2 + 3 = (2)^3 + 5(2)^2 + 3 = 31$

In doing this we are really using a bunch of the limits laws.

$\lim_{x\to 2} x^3 + 5x^2 + 3$

$= \lim_{x\to 2} x^3 + \lim_{x\to 2} 5x^2 + \lim_{x\to 2} 3$

$= [\lim_{x\to 2} x]^3 + 5[\lim_{x\to 2} x]^2 + 3$

$= (2)^3 + 5(2)^2 + 3 = 31$

ex.2 $\lim\limits_{x\to -1} \dfrac{2x^2-1}{x+3} = \dfrac{2(-1)^2-1}{-1+3} = \dfrac{1}{2}$

Here is how we are using the limit laws in this problem:

$\lim\limits_{x\to -1} \dfrac{2x^2-1}{x+3}$

$= \dfrac{\lim\limits_{x\to -1} 2x^2-1}{\lim\limits_{x\to -1} x+3}$

$= \dfrac{2[\lim\limits_{x\to -1} x]^2 - \lim\limits_{x\to -1} 1}{\lim\limits_{x\to -1} x + \lim\limits_{x\to -1} 3}$

$= \dfrac{2(-1)^2-1}{-1+3} = \dfrac{1}{2}$

Eliminating a denominator whose limit is zero

ex.3 Determine the following limit: $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$

When we try to use substitution we get

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{(2)^2-4}{2-2} = \frac{0}{0}$$

which is **undefined**.

Instead, we use algebra to eliminate the denominator:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x+2 = 4$$

ex.4 Determine the following limit: $\lim_{x \to -3} \frac{x^2 + 9x + 18}{x^2 - x - 12}$

When we try to use substitution we get $\lim_{x \to -3} \frac{x^2 + 9x + 18}{x^2 - x - 12} = \frac{0}{0}$ which is again **undefined**.

Use algebra:

$$\lim_{x \to -3} \frac{x^2 + 9x + 18}{x^2 - x - 12}$$

$$= \lim_{x \to -3} \frac{(x+3)(x+6)}{(x+3)(x-4)}$$

$$= \lim_{x \to -3} \frac{x+6}{x-4} = \frac{-3+6}{-3-4} = -\frac{3}{7}$$

ex.5 Determine the following limit: $\lim_{h \to 0} \frac{\sqrt{h+2} - \sqrt{2}}{h}$

We will multiply by the conjugate of the numerator to get,

$$\lim_{h \to 0} \frac{\sqrt{h+2} - \sqrt{2}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{h+2} - \sqrt{2}}{h} \cdot \frac{\sqrt{h+2} + \sqrt{2}}{\sqrt{h+2} + \sqrt{2}}$$

$$= \lim_{h \to 0} \frac{h + 2 - 2}{h(\sqrt{h+2} + \sqrt{2})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{h+2} + \sqrt{2}}$$

$$= \frac{1}{\sqrt{0+2} + \sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}}$$