

5.3. The Definite Integral

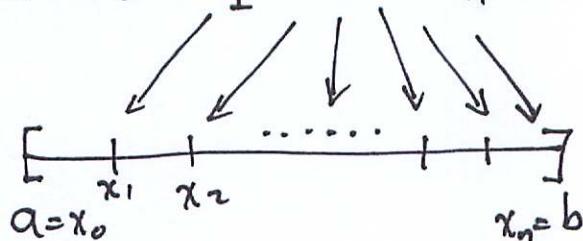
* The Set-Up

- A function $y = f(x)$ on an interval $[a, b]$.
- All possible ways of chopping $[a, b]$ into any number n of pieces.

This is called a partition

$$P = \{x_0, x_1, \dots, x_n\}$$

where $a = x_0 < x_1 < \dots < x_n = b$



- All ways of choosing "sample points":

c_1 in 1st interval $[x_0, x_1]$

c_2 in 2nd interval $[x_1, x_2]$

\dots
 c_n in nth (last) interval $[x_{n-1}, x_n]$.

- The narrow lengths: $\Delta x_1 = x_1 - x_0$, $\Delta x_2 = x_2 - x_1$,

$$\dots, \boxed{\Delta x_k = x_k - x_{k-1}}, \dots \Delta x_n = x_n - x_{n-1}$$

of each subinterval.

Note: If all intervals equal:
 $\Delta x_k = \frac{b-a}{n}$.

The Definite Integral

* The Riemann Sums

$$\sum_{k=1}^n f(c_k) \Delta x_k.$$

(Note: Perhaps $f(x)$ takes positive or negative values, or both, so the Riemann sum may not have a direct interpretation as an area, because $f(c_k) \Delta x_k$ may be negative, but "negative areas" don't exist!)

* The NORM of any partition,

$$\mathcal{P} = \{x_0, x_1, \dots, x_n\}$$

is denoted by $\|\mathcal{P}\|$ and means:

$$\|\mathcal{P}\| = \text{maximum of all } \Delta x_k$$

= maximum of the numbers
 $x_1 - x_0$ & $x_2 - x_1$ & ... & $x_n - x_{n-1}$.

Small norm means fine chopping of $[a, b]$.

5.3...

(3)

Definition:

If the limit of the Riemann sums

$$\sum_{k=1}^n f(c_k) \Delta x_k$$

exists when P ~~is any~~ varies over all possible partitions, and $\|P\| \rightarrow 0$

We say that the value of this limit is the **definite integral** of $f(x)$ over $[a, b]$.

Definite integral of $f(x)$ over $[a, b]$

$$= \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k.$$

Remark:

This is quite a difficult limit to verify as stated.

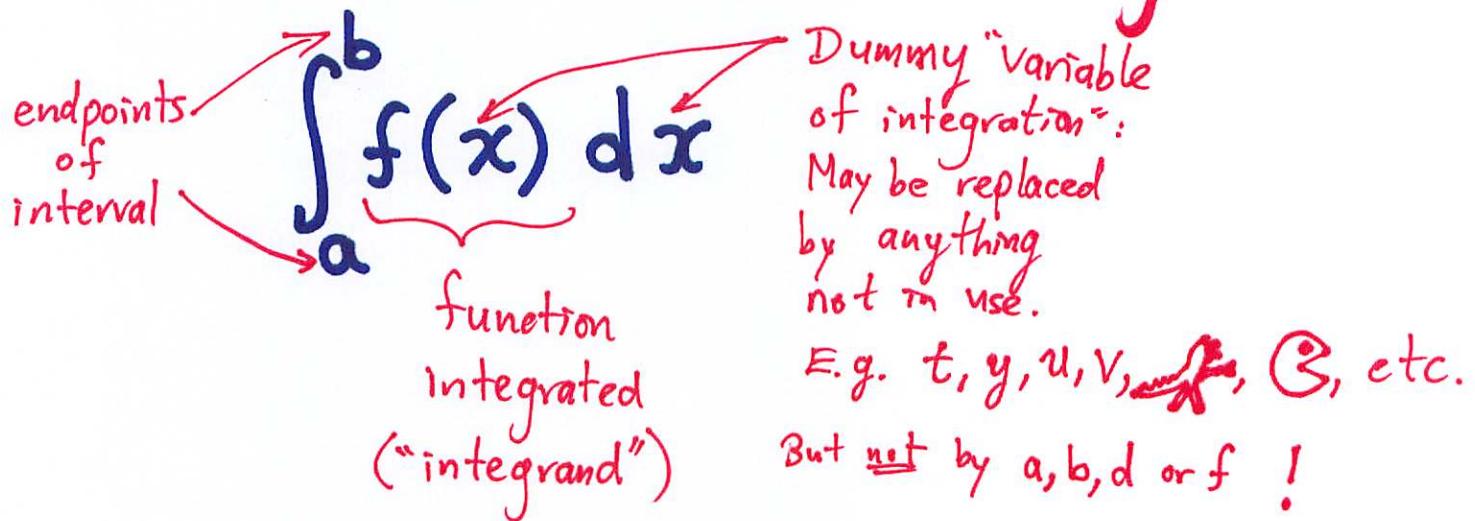
In practice, we will always take uniform partitions

$$P = \{x_0, x_1, \dots, x_n\}$$

where all Δx_k are equal: $\Delta x = \frac{b-a}{n} = \|P\|$.

Then $\|P\| \rightarrow 0$ Just means $n \rightarrow \infty$.

Notation for Definite Integrals



The "integral sign" \int (an extremely elongated "S", suggesting a kind of "sum")

and the "differential" letter d are fixed and mandatory.

The above is read:

"The definite integral of $f(x)$ "dee" x
from a to b ".

Vital Remark:

$\int_a^b f(x) dx$ is a number (the limit of Riemann sums).

It is definitely not a formula, and much less so a formula involving x (which is purely fictitious anyway, and has no specific value!).

5.3Example:

$$\boxed{\int_0^1 (1-x^2) dx = \frac{2}{3}}$$

(remember this?)

Theorem: If $f(x)$ is continuous on $[a,b]$ then $\int_a^b f(x) dx$ exists as a unique, well-defined number.

The definite integral of a discontinuous function may or may not exist. Just keep it in mind.

Limit-Swapping Convention.

We have agreed in writing $\int_a^b f(x) dx$ that we have $a \leq b$ with a on bottom & b on top.

To allow more flexibility, we allow the opposite case provided we interpret the integral as changing sign:

$$\int_b^a f(x) dx = - \int_a^b f(x) dx.$$

Example: $\int_1^0 (1-x^2) dx = -\frac{2}{3}.$

5.3...

Rules for Definite Integrals:

- $\int_a^a f(x) dx = 0$ (An integral over a null-length interval $[a, a]$ is zero)

- $\int_a^b c \cdot f(x) dx = c \cdot \int_a^b f(x) dx$ (Any constant factor c pulls out of the integral.)

- $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

(The integral of a sum/difference is equal to the sum/difference of the integrals.)

- $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

(The integrals over intervals sharing an endpoint add up to the integral over a single interval.

Note that we are not assuming a, b, c to be ordered $a < b < c$. We could have $a=7, b=0$ & $c=3$, say).

NON-rules:

- $\int_a^b f(x) g(x) dx$ NOT same as $\int_a^b f(x) dx \cdot \int_a^b g(x) dx$

- $\int_a^b \frac{f(x)}{g(x)} dx$ NOT same as $\int_a^b f(x) dx / \int_a^b g(x) dx$

- $\int_a^b (f(x))^2 dx$ NOT same as $\left(\int_a^b f(x) dx \right)^2$, ETC.

5.3...

(7)

Monotonicity Properties of \int_a^b :

Min-Max inequality.

If $m \leq f(x) \leq M$ on $[a, b]$

(perhaps, say,
 $m = \text{minimum of } f$
 $M = \text{maximum of } f$)

then $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$.

Another way to think of this is through averages:

The average of $f(x)$ on $[a, b]$ is defined as:

$$f_{\text{av}} = \frac{1}{b-a} \int_a^b f(x) dx$$

The Min-Max inequality says that

$$m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M$$

Same, see?

In other words, f_{av} lies somewhere between the minimum and maximum values of $f(x)$ on $[a, b]$

Domination.

If $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$

If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x) dx \geq 0$.
"g(x)"

Clearly $0 = \int_a^b 0 dx$

5.3 ...

(8)

Example: Show that $\int_{-\pi}^{\pi} \sqrt{17+8\cos x} dx$

is between 6π and 10π .

Clearly $f(x) = \sqrt{17+8\cos x}$ has minimum value

$$\text{minimum value } m = \sqrt{17-8} = \sqrt{9} = 3,$$

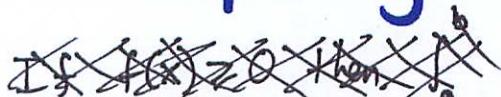
$$\text{maximum value } M = \sqrt{17+8} = \sqrt{25} = 5.$$

By Min-Max inequality:

$$6\pi = 3 \cdot 2\pi \leq \int_{-\pi}^{\pi} \sqrt{17+8\cos x} dx \leq 5 \cdot 2\pi = 10\pi$$

min m length of $[-\pi, \pi]$ max M length of $[-\pi, \pi]$

Interpreting $\int_a^b f(x) dx$ using areas.



In intervals $[a, r]$ and $[s, b]$

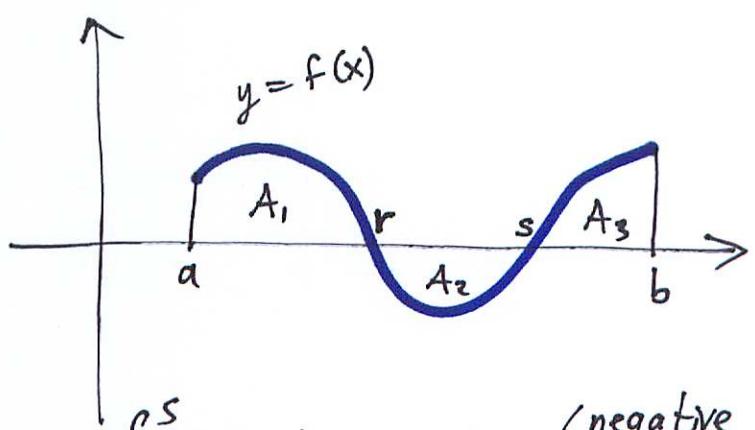
we have $f(x) \geq 0$:

$$\int_a^r f(x) dx = A_1 \text{ (area)}$$

$$\int_s^b f(x) dx = A_3 \text{ (area)}$$

But in $[r, s]$ we have $f(x) \leq 0$: $\int_r^s f(x) dx = -A_2$ (^{negative} of area)

$$\text{Altogether: } \int_a^b f(x) dx = \int_a^r f(x) dx + \int_r^s f(x) dx + \int_s^b f(x) dx = A_1 - A_2 + A_3.$$



53...

$$\underline{\text{Ex:}} \int_a^b dx = \int_a^b 1 \cdot dx = ?$$

Riemann sum $\sum_{k=1}^n 1 \cdot \Delta x_k = \sum_{k=1}^n \Delta x_k = b-a$

(the sum of the lengths of all subintervals
is the length $b-a$ of the whole interval.

If all the intervals were equal, $\Delta x = \frac{b-a}{n}$
and $\sum_{k=1}^n \Delta x = n \cdot \Delta x = (b-a)$.

Taking the limit $\|\mathcal{P}\| \rightarrow 0$ or $n \rightarrow \infty$
affects nothing, so:

$$\int_a^b dx = \int_a^b 1 dx = b-a.$$

5.3. ..

(10)

$$\underline{\text{EX}}: \int_0^b x dx$$

"f(x)"

Partition $[0, b]$ into n equal parts

$$P = \left\{ 0 = x_0, x_1 = \frac{b}{n}, x_2 = \frac{2b}{n}, \dots, \boxed{x_k = k \frac{b}{n}}, \dots, x_n = b \right\}.$$

For simplicity we take $c_k = x_k$ (right endpoint

Also, $\Delta x = \frac{b-0}{n} = \frac{b}{n}$. of $[x_{k-1}, x_k]$)

Riemann Sum:

$$\sum_{k=1}^n c_k \cdot \Delta x = \sum_{k=1}^n \left(k \frac{b}{n} \right) \cdot \frac{b}{n} = \sum_{k=1}^n \frac{b^2}{n^2} k$$

constant independent
of summation
variable k .
(pulls out)

$$\begin{aligned} &= \frac{b^2}{n^2} \sum_{k=1}^n k = \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} \\ &= \frac{b^2}{2} \cdot \frac{n+1}{n} = \frac{b^2}{2} (1 + \gamma_n) \end{aligned}$$

(Formula
for sum
of consecutive
integers.)

$$\int_0^b x dx = \lim_{n \rightarrow \infty} \frac{b^2}{2} (1 + \gamma_n) = \frac{b^2}{2} (1 + 0) = \frac{b^2}{2}.$$

$$\int_0^b x dx = \frac{b^2}{2}$$

$\text{Also: } \int_a^b x dx = \int_a^0 x dx + \int_0^b x dx$ $= - \int_0^a x dx + \int_0^b x dx$ $= -\frac{a^2}{2} + \frac{b^2}{2}.$
