The Derivative as a function

Previously we used the limit of the slopes of the secant lines through the points $(a, f(a))$ and $(a+h, f(a+h))$ as $h \to 0$ to find the value of the derivative of $f(x)$ at $x = a$. We will now apply this idea to the function $f(x)$ for each x in its domain and define a new function,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to be the derivative of the function $f(x)$ at x.

Here, x is any number in the domain of $f(x)$ and the derivative, $f'(x)$, exists whenever the above limit exists.

ex.1 Use the definition of the derivative to find $f'(x)$ given the function, $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

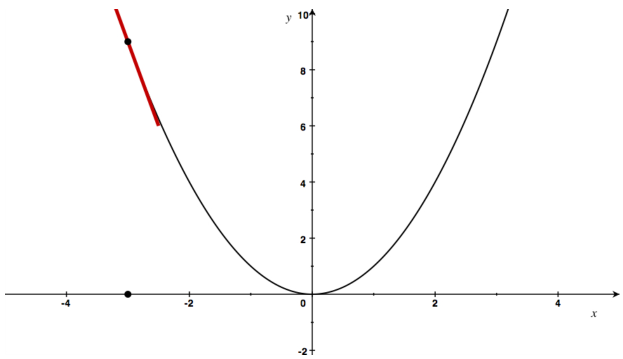
$$= \lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$= \lim_{h \to 0} 2x + h$$

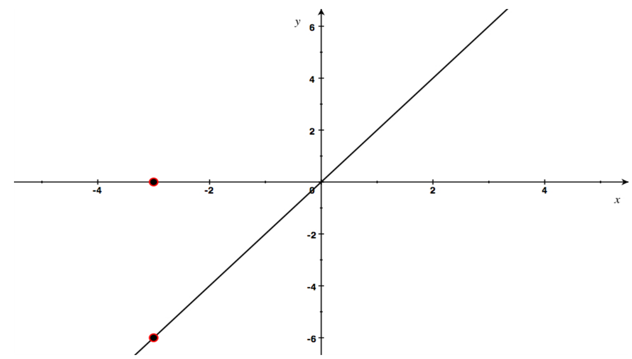
$$= 2x + 0$$

$$= 2x$$

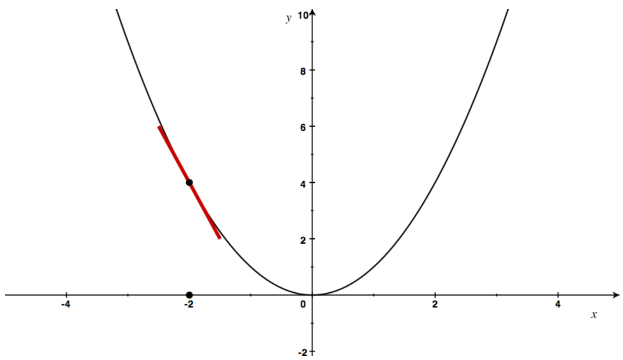
The following graphs illustrate the connection between the function $f(x) = x^2$ and its derivative function $f'(x) = 2x$



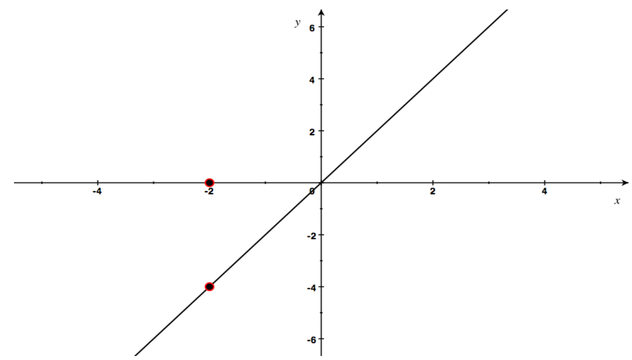
Slope of tangent line at $x = -3$ is $f'(-3) = -6$



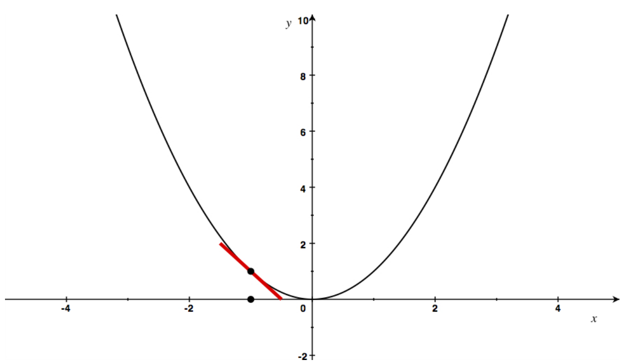
Value of derivative at $x = -3$ is -6



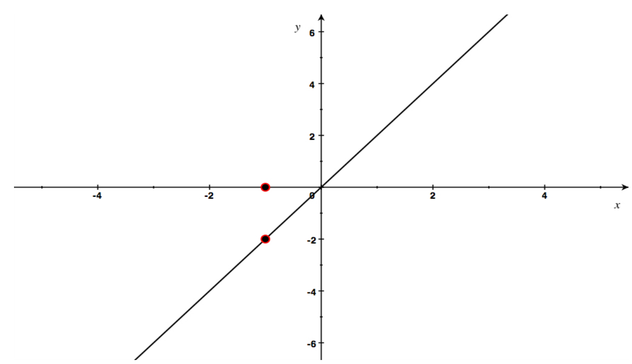
Slope of tangent line at $x = -2$ is $f'(-2) = -4$



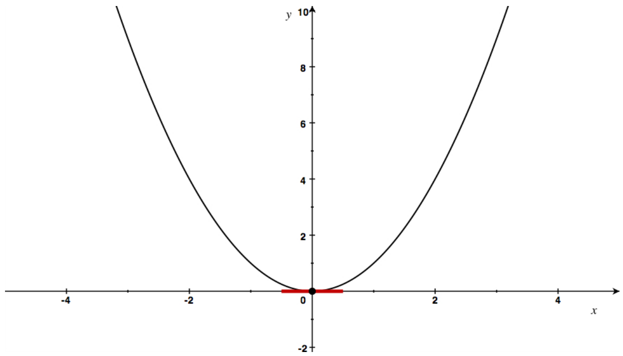
Value of derivative at $x = -2$ is -4



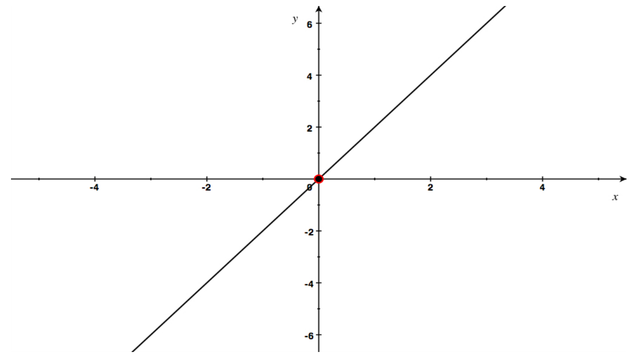
Slope of tangent line at $x = -1$ is $f'(-1) = -2$



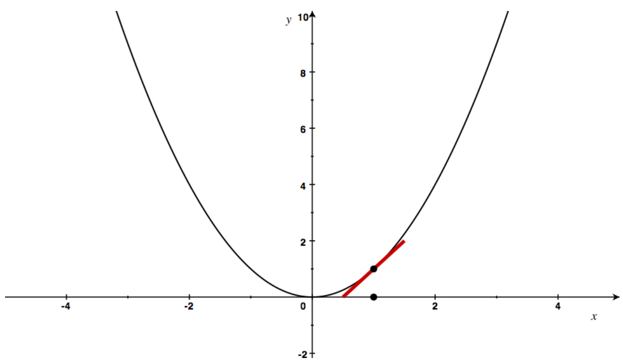
Value of derivative at $x = -1$ is -2



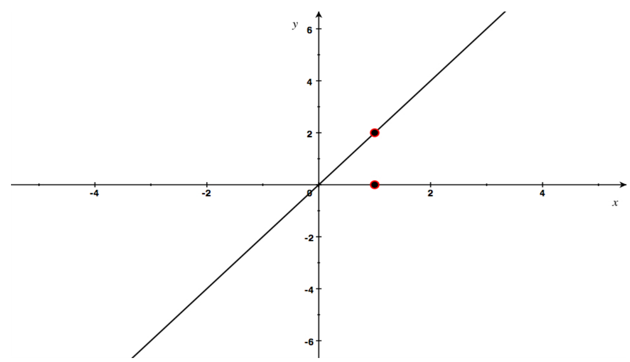
Slope of tangent line at $x = 0$ is $f'(0) = 0$



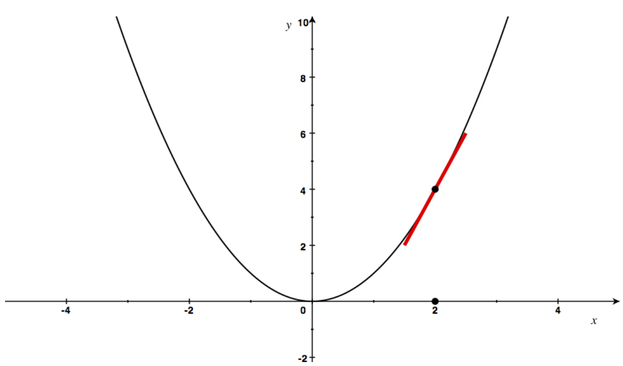
Value of derivative at $x = 0$ is 0



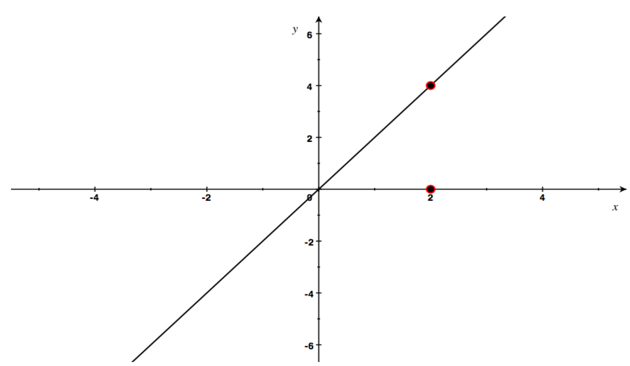
Slope of tangent line at $x = 1$ is $f'(1) = 2$



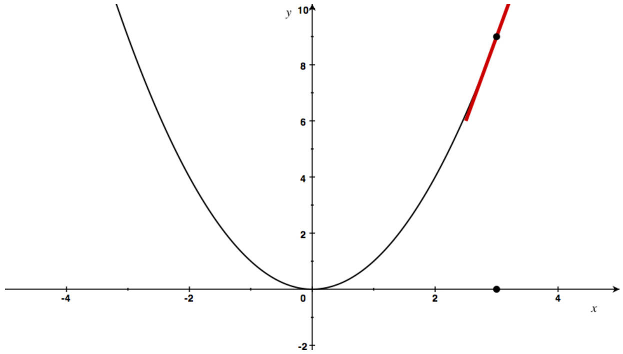
Value of derivative at $x = 1$ is 2



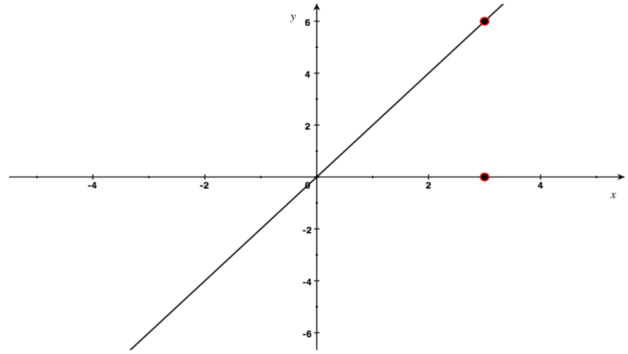
Slope of tangent line at $x = 2$ is $f'(2) = 4$



Value of derivative at $x = 2$ is 4



Slope of tangent line at $x = 3$ is $f'(3) = 6$



Value of derivative at $x = 3$ is 6

We have seen that the derivative of a function at a point, the instantaneous rate of change of a function at a point, and the slope of the tangent line to the curve of a function at a point are **all the same thing**.

Left and Right Derivatives

A function is differentiable on a closed interval $[a, b]$ if it is differentiable on (a, b) and the following limits extist:

Left Hand Derivative at $x = a$: $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$

Right Hand Derivative at $x = b$: $\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$

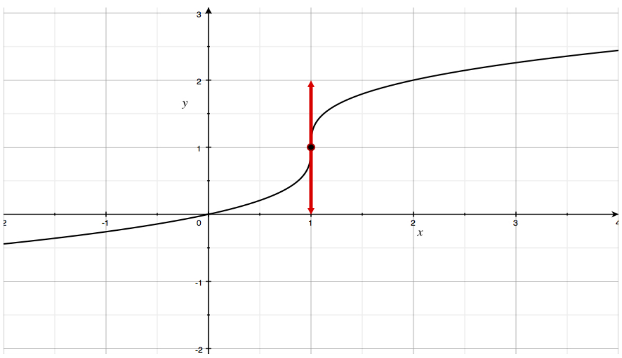
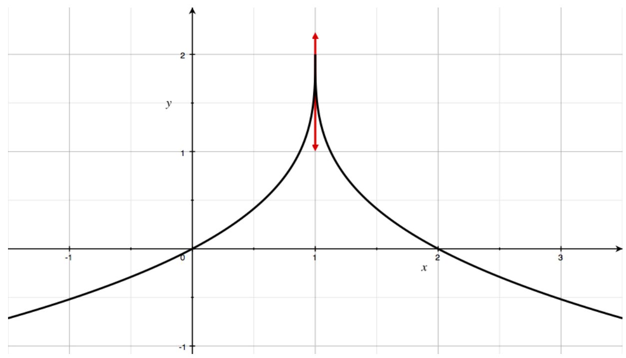
If x is any number in the open interval (a, b) then the derivative $f'(x)$ exists if and only if the left and right hand derivatives at x exist and are equal.

Situations where the derivative fails to exist

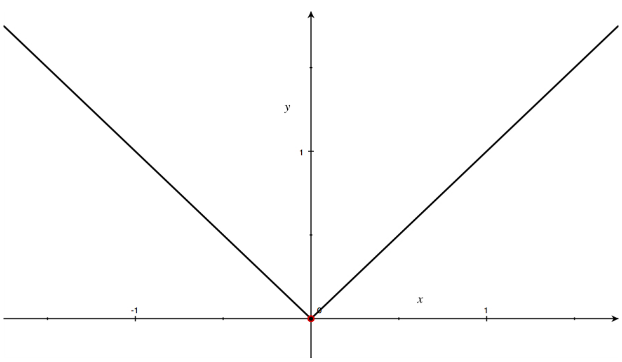
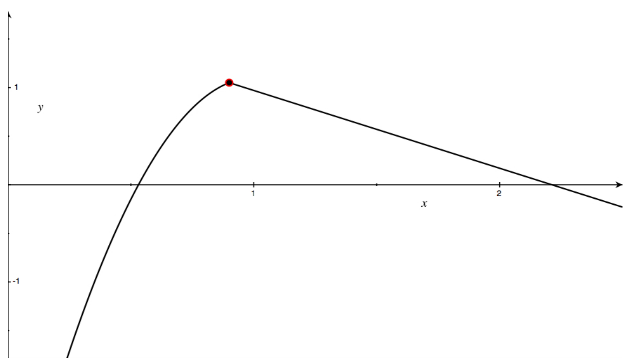
ex.1 Function has a jump or removable discontinuity. When this happens, the left and right derivatives are not equal or the function is not defined at a point. When the function is defined at a point but the limit of the function and its value are not equal, Say we have $f(x) = x^2$, $x \neq 1$ and $f(x) = -1$ for $x = 1$ then,

$$f'(x=1) = \lim_{h \to 0} \frac{(1+h)^2 - (-1)}{h} = \lim_{h \to 0} \frac{1 + 2h + h^2 + 1}{h} = \lim_{h \to 0} \frac{h^2 + 2h + 2}{h} \text{ which does not exist.}$$

ex.2 Tangent line is vertical in which case its slope is undefined.

ex.3. At a corner point. The left and right derivatives are not equal.

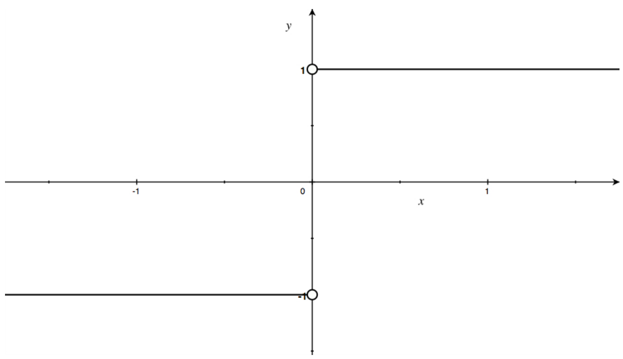
Above left is the graph of the absolute value function, $f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

Notice that to the left of zero, $x < 0$, the derivative of f has a value of -1 and to the right of zero the derivative has a value of 1. The left and right hand derivatives are not equal at zero and so the derivative does not exist at zero.

Using this information, we can define the derivative of $f(x)$ to be $f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$

Notice that this definition provides no value at zero since f' is not defined there.

Here is the graph:



Using the definition of derivative and $|x| = \sqrt{x^2}$ as our definition of absolute value function we arrive at the same derivative function, but a little different looking:

The derivative of $f(x) = |x|$ is $f'(x) = \dfrac{x}{|x|}$